A REMARK ON GLOBAL WELL-POSEDNESS OF THE DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION ON THE CIRCLE

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Abstract. In this note, we consider the derivative nonlinear Schrödinger equation on the circle. In particular, by adapting Wu's recent argument to the periodic setting, we prove its global well-posedness in $H^1(\mathbb{T})$, provided that the mass is less than 4π. Moreover, this mass threshold is independent of spatial periods.

1. Introduction

In this note, we consider global well-posedness of the following derivative nonlinear Schrödinger equation (DNLS) on $\mathbb{T}_L := \mathbb{R}/(L\mathbb{Z}) \simeq [0, L)$:

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\partial_x(|u|^2 u) \\ u|_{t=0} = u_0 \in H^1(\mathbb{T}_L), \end{cases} \qquad (x,t) \in \mathbb{T}_L \times \mathbb{R}. \tag{1.1}$$

The equation (1.1) is known to be completely integrable and thus possesses an infinite sequence of conservation laws. For our analysis, the following conservation laws play an important role:

$$\text{Mass:} \qquad M(u) = \int_{\mathbb{T}_L} |u|^2 dx, \tag{1.2}$$

$$\text{Hamiltonian:} \qquad H(u) = \operatorname{Im} \int_{\mathbb{T}_L} u\overline{u}_x dx + \frac{1}{2}\int_{\mathbb{T}_L} |u|^4 dx, \tag{1.3}$$

$$\text{Energy:} \qquad E(u) = \int_{\mathbb{T}_L} |u_x|^2 dx + \frac{3}{2}\operatorname{Im}\int_{\mathbb{T}_L} uu\overline{u}\overline{u}_x dx + \frac{1}{2}\int_{\mathbb{T}_L} |u|^6 dx. \tag{1.4}$$

Let us briefly go over the known well-posedness results on $\mathbb{T}$, i.e. with $L = 1$. Herr [5] proved local well-posedness of (1.1) in $H^{\frac{1}{2}}(\mathbb{T})$. He also proved global well-posedness in $H^1(\mathbb{T})$, under the assumption that the mass is less than $\frac{2}{3}$.[1] In the low regularity setting, Win [10] applied the I-method [2, 3] and proved global well-posedness of (1.1) in $H^s(\mathbb{T})$, $s > \frac{1}{2}$, provided that mass is sufficiently small.[2] Our main interest in this note is to improve the mass threshold for global well-posedness of (1.1) in the smooth setting, i.e. in $H^1(\mathbb{T}_L)$.

On $\mathbb{R}$, Hayashi-Ozawa [4] proved global well-posedness of (1.1) in $H^1(\mathbb{R})$, provided that mass is less than 2π. By the sharp Gagliardo-Nirenberg inequality due to Weinstein [9]:

$$\|f\|_{L^6(\mathbb{R})} \le \frac{4}{\pi^2}\|\partial_x f\|_{L^2(\mathbb{R})}^{\frac{1}{3}}\|f\|_{L^2(\mathbb{R})}^{\frac{2}{3}},$$

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1. As pointed out in [5, Remark 6.1], this mass threshold $\frac{2}{3}$ is not sharp. In view of the corresponding result [4] on $\mathbb{R}$, it is likely that the mass threshold can be improved to 2π within the framework of [5].

2. In [10], the mass threshold was not quantified in a precise manner. See, for example, [10, Lemma 3.4].

this smallness of mass guarantees that the energy $E(u)$ remains coercive and controls the $\dot{H}^1(\mathbb{R})$-norm of a solution. Thus, this situation is analogous to that for the focusing quintic nonlinear Schrödinger equation (NLS). [3] On the one hand, there is a dichotomy between global well-posedness and finite time blowup solutions for the focusing quintic NLS on $\mathbb{R}$, where the mass threshold is given by the mass of the ground state. On the other hand, DNLS has a much richer structure such as complete integrability and the question of global well-posedness/finite time blowup solutions for large masses has been open for decades. Recently, Wu [11, 12] made a progress in this direction. In particular, he proved global well-posedness of (1.1) on $\mathbb{R}$ for masses less than 4π. Our main result states that global well-posedness of (1.1) in the periodic setting also holds with the same mass threshold 4π.

Theorem 1.1. *Let $L > 0$. Then, the derivative nonlinear Schrödinger equation* (1.1) *on $\mathbb{T}_L$ is globally well-posed in $H^1(\mathbb{T}_L)$, provided that the mass is less than 4π.*

Theorem 1.1 improves the known mass threshold in [5] for global well-posedness in $H^1(\mathbb{T})$. Moreover, note that the mass threshold 4π is independent of the period L.

The question of global well-posedness/finite time blowup solutions for larger masses ($\geq 4\pi$) remains open on both $\mathbb{R}$ and $\mathbb{T}_L$. It is worthwhile to note that (1.1) possesses finite time blowup solutions under the Dirichlet boundary condition on intervals and the half line $\mathbb{R}_+ = [0, \infty)$, if $E(u) < 0$ (under some extra conditions). See [8, 11].

The proof of Theorem 1.1 is based on Wu's argument [12]. On the one hand, the following sharp Gagliardo-Nirenberg inequality:

$$\|f\|_{L^6(\mathbb{R})} \leq C_{\mathrm{GN}} \|\partial_x f\|_{L^2(\mathbb{R})}^{\frac{1}{9}} \|f\|_{L^4(\mathbb{R})}^{\frac{8}{9}} \tag{1.5}$$

plays an important role in [12]. Here, the optimal constant C_{GN} is given by $C_{\mathrm{GN}} = 3^{\frac{1}{6}}(2\pi)^{-\frac{1}{9}}$. See Agueh [1]. On the other hand, (1.5) does not hold on $\mathbb{T}_L$ and thus we need to consider a variation of (1.5) suitable for our application on $\mathbb{T}_L$. Moreover, the gauge transform in the periodic setting introduces extra terms in the conservation laws that we need to control.

2. Proof of Theorem 1.1

In this section, we present the proof of Theorem 1.1. Note that Theorem 1.1 follows once we prove the following proposition for all sufficiently small $\delta > 0$.

Proposition 2.1. *Let $L, \delta > 0$. Then,* (1.1) *on $\mathbb{T}_L$ is globally well-posed in $H^1(\mathbb{T}_L)$ provided that the mass is less than $4\pi\big(1 + \frac{2\delta}{5L}\big)^{-2}$.*

The remaining part of this note is devoted to the proof of Proposition 2.1.

We first establish the following version of the Gagliardo-Nirenberg inequality on $\mathbb{T}_L$ which incorporates the sharp constant from (1.5). The proof is a simple adaptation of the argument in Lebowitz-Rose-Speer [6].

Lemma 2.2. *Let $\delta > 0$. Then, we have*

$$\|f\|_{L^6(\mathbb{T}_L)} \leq C_{\mathrm{GN}} \bigg(1 + \frac{2\delta}{5L}\bigg)^{\frac{2}{9}} \bigg(\|\partial_x f\|_{L^2(\mathbb{T}_L)}^2 + \frac{2}{\delta L^{\frac{1}{2}}} \|f\|_{L^4(\mathbb{T}_L)}^2\bigg)^{\frac{1}{18}} \|f\|_{L^4(\mathbb{T}_L)}^{\frac{8}{9}}. \tag{2.1}$$

for $f \in H^1(\mathbb{T}_L)$.

3. Note that both DNLS and the focusing quintic NLS on $\mathbb{R}$ are mass-critical.

Proof. Let $f \in H^1(\mathbb{T}_L) \subset C(\mathbb{T}_L)$. By periodicity, we assume that

$$|f(0)| = |f(L)| \leq L^{-\frac{1}{4}} \|f\|_{L^4(\mathbb{T}_L)} \tag{2.2}$$

without loss of generality. Let F be an extension of f on $[0, L]$ to $\mathbb{R}$ such that (i) $\operatorname{supp} F \subset [-\delta, L+\delta]$ and (ii) F linearly interpolates 0 and $f(0)$ on $[-\delta, 0]$ and $f(L)$ and 0 on $[L, L+\delta]$. Then, by a direct calculation, we have

$$\|f\|^6_{L^6(\mathbb{T}_L)} \leq \|F\|^6_{L^6(\mathbb{R})}, \tag{2.3}$$

$$\|F\|^4_{L^4(\mathbb{R})} \leq \|f\|^4_{L^4(\mathbb{T}_L)} + \frac{2\delta}{5}|f(0)|^4 \leq \left(1 + \frac{2\delta}{5L}\right)\|f\|^4_{L^4(\mathbb{T}_L)}, \tag{2.4}$$

$$\|\partial_x F\|^2_{L^2(\mathbb{R})} \leq \|\partial_x f\|^2_{L^2(\mathbb{T}_L)} + 2\frac{|f(0)|^2}{\delta} \leq \|\partial_x f\|^2_{L^2(\mathbb{T}_L)} + \frac{2}{\delta L^{\frac{1}{2}}}\|f\|^2_{L^4(\mathbb{T}_L)}. \tag{2.5}$$

Then, the desired estimate (2.1) follows from (1.5) with (2.3), (2.4), and (2.5). $\square$

Next, we briefly go over the gauge transform associated to (1.1) with a general parameter $\beta \in \mathbb{R}$. The gauge transform for DNLS was first introduced by Hayashi-Ozawa [4] in the non-periodic setting. Herr [5] adapted the gauge transform (with $\beta = 1$) to the periodic setting, exhibiting remarkable cancellations of certain resonances.

Given $f \in H^1(\mathbb{T}_L)$, let $\mathcal{I}(f)$ denotes the mean-zero antiderivative of $|f|^2$. Then, we define $\mathcal{G}_\beta : H^1(\mathbb{T}_L) \to H^1(\mathbb{T}_L)$ by $\mathcal{G}_\beta(f) := e^{-i\beta\mathcal{I}(f)} f$. With a slight abuse of notations, we also use $\mathcal{G}_\beta$ to denote a map: $C([-T, T] : H^1(\mathbb{T}_L)) \to C([-T, T] : H^1(\mathbb{T}_L))$ by

$$\mathcal{G}_\beta(u) := e^{-i\beta\mathcal{I}(u)} u.$$

Given a local-in-time solution $u \in C([-T, T] : H^1(\mathbb{T}_L))$ to (1.1), the conservation of mass allows us to define

$$\mu = \mu(u) := \frac{1}{L} M(u) = \frac{1}{L}\int_{\mathbb{T}_L} |u|^2 dx,$$

independent of time. We then define

$$v(x, t) := \mathcal{G}^\beta(u)(x, t) = \mathcal{G}_\beta(u)(x - 2\beta\mu t, t), \tag{2.6}$$

A straightforward computation shows that v satisfies

$$i\partial_t v + \partial_x^2 v = 2(1-\beta)i|v|^2 v_x + (1 - 2\beta)iv^2\overline{v}_x + \beta\mu|v|^2 v + \beta(\tfrac{1}{2} - \beta)|v|^4 v - \psi(v)v, \tag{2.7}$$

where

$$\psi(v) := \frac{\beta}{L}\left(\int_{\mathbb{T}_L} 2\operatorname{Im}(v\overline{v}_x) + \left(\frac{3}{2} - 2\beta\right)|v|^4\right)v + \beta^2\mu^2.$$

It follows from (2.6) that $M(v)$ is conserved for (2.7). Moreover, the conservation laws $H(u)$ and $E(u)$ in (1.3) and (1.4) for (1.1) yield the following conservation laws for (2.7):

$$H(v) = \operatorname{Im}\int_{\mathbb{T}_L} v\overline{v}_x dx + \left(\frac{1}{2} - \beta\right)\int_{\mathbb{T}_L} |v|^4 dx + L\beta\mu^2, \tag{2.8}$$

$$\begin{aligned} E(v) = &\int_{\mathbb{T}_L} |v_x|^2 dx + \left(\frac{3}{2} - 2\beta\right)\operatorname{Im}\int_{\mathbb{T}_L} vv\overline{v}\overline{v}_x dx + \left(\beta^2 - \frac{3}{2}\beta + \frac{1}{2}\right)\int_{\mathbb{T}_L} |v|^6 dx \\ &+ 2\beta\operatorname{Im}\int_{\mathbb{T}_L} v\overline{v}_x dx + \beta\left(\frac{3}{2} - 2\beta\right)\mu\int_{\mathbb{T}_L} |v|^4 dx + L\beta^2\mu^3. \end{aligned} \tag{2.9}$$

See, for example, the computations in [7]. It is worthwhile to note that $H(v)$ is not a Hamiltonian for (2.7) in general. In establishing well-posedness, the gauge transform with $\beta = 1$ played an important role [4, 5, 10]. For our purpose, we set $\beta = \frac{3}{4}$ in the following so

that the second term in (2.9) is not present, and let $\mathcal{G} := \mathcal{G}^{\frac{3}{4}}$. In particular, it follows from (2.8) and (2.9) with the conservation of $\mu = \mu(v) := L^{-1}M(v)$ that the following quantity

$$\mathcal{E}(v) := \int_{\mathbb{T}_L} |v_x|^2 dx - \frac{1}{16}\int_{\mathbb{T}_L} |v|^6 dx + \frac{3}{8}\mu \int_{\mathbb{T}_L} |v|^4 dx. \tag{2.10}$$

is conserved for (2.7), where $v = \mathcal{G}(u)$.

Now, we move onto the proof of Proposition 2.1. The proof follows closely to that in [12]. By time reversibility, we restrict our attention to positive times. For notational simplicity, we suppress the domain of integration $\mathbb{T}_L$ with the understanding that all the norms are taken over $\mathbb{T}_L$. First, recall that Herr's local well-posedness result [5] yields a simple blowup alternative; either (i) the solution u to (1.1) exists globally or (ii) there exists a finite time T_* such that $\lim_{t\uparrow T_*} \|u(t)\|_{\dot H^1} = \infty$.

Fix $\delta > 0$. We argue by contradiction. Suppose that there exists a solution u to (1.1) such that $M(u) < 4\pi\big(1 + \frac{2\delta}{5L}\big)^{-2}$ but $\lim_{t\uparrow T_*} \|u(t)\|_{\dot H^1} = \infty$ for some finite time $T_* > 0$. Let $v = \mathcal{G}(u)$ be the corresponding solution to (2.7). Since the gauge transform $\mathcal{G}$ in (2.6) is continuous on $C([-T,T] : H^1)$, our assumption implies that there exists a sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that $\lim_{n\to\infty} \|v(t_n)\|_{\dot H^1} = \infty$ while $M(v) = M(u) < 4\pi\big(1 + \frac{2\delta}{5L}\big)^{-2}$. Then, it follows from the conservation of $\mathcal{E}(v)$ that

$$\|v(t_n)\|_{L^6} \to \infty, \tag{2.11}$$

as $n \to \infty$.

As in [12], we define $\{f_n\}_{n\in\mathbb{N}}$ by

$$f_n = \frac{\|v(t_n)\|_{L^4}^4}{\|v(t_n)\|_{L^6}^3}.$$

Then, we have the following lemma.

Lemma 2.3. *Let $L, \delta > 0$. Then, we have*

$$2C_{\mathrm{GN}}^{-\frac{9}{2}}\bigg(1 + \frac{2\delta}{5L}\bigg)^{-1} + \varepsilon_n \le f_n \le M(v)^{\frac{1}{2}}, \tag{2.12}$$

where $\varepsilon_n = \varepsilon_n(L,\delta) \to 0$ as $n \to \infty$. In particular, $\|v(t_n)\|_{L^4} \to \infty$ as $n \to \infty$.

Proof. The upper bound in (2.12) follows from Hölder's inequality. Then, it follows from the upper bound in (2.12) and (2.11) that

$$\gamma_n := \bigg(\frac{2}{\delta L^{\frac{1}{2}}} - \frac{3}{8}\mu\|v(t_n)\|_{L^4}^2\bigg)\frac{\|v(t_n)\|_{L^4}^2}{\|v(t_n)\|_{L^6}^6} \longrightarrow 0, \tag{2.13}$$

as $n \to \infty$. By Lemma 2.2 with (2.10), we have

$$\begin{aligned} f_n &\ge C_{\mathrm{GN}}^{-\frac{9}{2}}\bigg(1 + \frac{2\delta}{5L}\bigg)^{-1}\bigg(\|\partial_x v(t_n)\|_{L^2}^2 + \frac{2}{\delta L^{\frac{1}{2}}}\|v(t_n)\|_{L^4}^2\bigg)^{-\frac{1}{4}}\|v(t_n)\|_{L^6}^{\frac{3}{2}} \\ &= 2C_{\mathrm{GN}}^{-\frac{9}{2}}\bigg(1 + \frac{2\delta}{5L}\bigg)^{-1}\bigg(1 + 16\frac{\mathcal{E}(v)}{\|v(t_n)\|_{L^6}^6} + 16\gamma_n\bigg)^{-\frac{1}{4}}. \end{aligned} \tag{2.14}$$

Then, the lower bound in (2.12) follows from (2.11), (2.13), and (2.14) with the conservation of $\mathcal{E}(v)$. The second claim follows from (2.11) and (2.12). $\square$

In the following, we use the conservation of the momentum $P(v)$ defined by

$$P(v) := H(v) - \frac{3}{4L}M(v)^2 = \operatorname{Im}\int_{\mathbb{T}_L} v\overline{v}_x dx - \frac{1}{4}\int_{\mathbb{T}_L} |v|^4 dx.$$

In order to exploit the momentum, we consider modulated functions $\phi_n(x,t) = e^{i\alpha_n x}v(x,t)$ for some non-zero $\alpha_n \in 2\pi\mathbb{Z}/L$ (to be chosen later). On the one hand, we have

$$P(v) + \frac{1}{4}\int_{\mathbb{T}_L} |v|^4 dx = \operatorname{Im}\int_{\mathbb{T}_L} v\overline{v}_x dx = -\frac{1}{2\alpha_n}\mathcal{E}(\phi_n) + \frac{\alpha_n}{2}M(v) + \frac{1}{2\alpha_n}\mathcal{E}(v). \tag{2.15}$$

On the other hand, by Lemma 2.2 with (2.10) and (2.13), we have

$$\mathcal{E}\big((\phi_n(t_n)\big) \geq -(\eta_n + \gamma_n)\|v(t_n)\|_{L^6}^6 \tag{2.16}$$

where η_n is defined by

$$\eta_n := \frac{1}{16} - \left(1 + \frac{2\delta}{5L}\right)^{-4} C_{\mathrm{GN}}^{-18} f_n^{-4}. \tag{2.17}$$

Case 1: $\eta_n + \gamma_n \leq 0$ for infinitely many n.

In this case, we simply set $\alpha_n = \frac{2\pi}{L}$. Then, for those values of n with $\eta_n + \gamma_n \leq 0$, it follows from (2.15) and (2.16) with (2.13) that

$$\begin{aligned}\frac{1}{4}\|v(t_n)\|_{L^4}^4 &\leq \frac{L}{4\pi}(\eta_n + \gamma_n)\|v(t_n)\|_{L^6}^6 - P(v) + \frac{\pi}{L}M(v) + \frac{L}{4\pi}\mathcal{E}(v)\\ &\leq -P(v) + \frac{\pi}{L}M(v) + \frac{L}{4\pi}\mathcal{E}(v).\end{aligned}$$

Then, from the conservation of M, P, and $\mathcal{E}$, we conclude that $\|v(t_n)\|_{L^4} = O(1)$. This is a contradiction to Lemma 2.3.

Case 2: $\eta_n + \gamma_n > 0$ for all sufficiently large n.

In this case, we choose

$$\alpha_n := \frac{2\pi}{L}\Big[\frac{L}{2\pi}\big(M(v)^{-1}(\eta_n + \gamma_n)\big)^{\frac{1}{2}}\|v(t_n)\|_{L^6}^3\Big] + \frac{2\pi}{L} \in \frac{2\pi\mathbb{Z}}{L},$$

where γ_n and η_n are as in (2.13) and (2.17). Here, $[x]$ denotes the integer part of x. Then, from (2.15) and (2.16), we have

$$\frac{1}{4}\|v(t_n)\|_{L^4}^4 \leq \big(M(v)(\eta_n + \gamma_n)\big)^{\frac{1}{2}}\|v(t_n)\|_{L^6}^3 - P(v) + \frac{\pi}{L}M(v) + \frac{1}{2\alpha_n}\mathcal{E}(v).$$

Then, by Lemma 2.3, (2.11), (2.13), and (2.17) along with the conservation of M, P, and $\mathcal{E}$, we obtain

$$f_n^6 \leq M(v)f_n^4 - 16\left(1 + \frac{2\delta}{5L}\right)^{-4} C_{\mathrm{GN}}^{-18}M(v) + o(1) \tag{2.18}$$

as $n \to \infty$. Arguing as in [12], we see that (2.18) is impossible if

$$M(u) = M(v) < 4\pi\left(1 + \frac{2\delta}{5L}\right)^{-2}.$$

This completes the proof of Proposition 2.1 and hence the proof of Theorem 1.1.

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